Vibration of pipelines under flexural dynamic loads

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A SYSTEM OF eight-coupled first-order partial differential equations describing the vibration response of pipelines under external flexural loads is derived. The decoupling of these equations yields a system of eight fourth-order partial differential equations. An analytical solution is achieved with the aid of integral transforms. Vibration analysis of pipelines subjected to impact and harmonic loads is provided.

Key words: vibration, pipelines, flexural loading, integral transforms

Pipelines are usually subjected to external flexural loads transmitted by pumps or compressors. Apart from the service loading, modern inspection technologies are based on the analysis of guided-wave propagation due to impact loading. Both impact and harmonic external load yield pipeline vibration.

An extended review provided by Tijsseling [1] presents the important models from the decade 1986-1996. An overview of newer techniques for dynamic analysis of pipelines can be obtained in the Refs 2-4. In these works, the method of characteristics (MOC) is used for time-domain solutions of pipelines under transient loading. Nowadays, dynamic analysis of pipelines is mainly carried out in the frequency domain [5]. In Ref.6, transfer matrices are used for the frequency response of curved or 3-D pipelines. A recent review of fluid pipe interaction models is provided in Ref.7. Tentarelli [8] and Liu [9] have systematically analysed the noise and vibration propagation in complex pipelines. Work on pipeline response under fluid-hammer conditions in steel and multi-layered anisotropic fibre-reinforced materials have recently published by the author [10-15].

In the present work an analytical solution of the eight-coupled differential equations for the dynamic response of pipelines subjected to flexural, impact, or harmonic external loads, is derived. The proposed methodology is based on double integral transforms.

Formulation of the model

An elementary pipe segment (Fig.1) is subjected to flexural dynamic loads per unit length $F_x(z,t)$ and $F_y(z,t)$, and bending moments per unit length $M_x(z,t)$ and $M_y(z,t)$. Taking into account inertia effects, the following motion equations in the planes $xz$ and $yz$ can be derived:

Flexure in $xz$ plane

Simulating the liquid-filled pipe with a beam, the transverse deflection $w_y$ due to bending stresses and shear are given [e.g. 16] as:

$$\phi_x + \phi_y = -\frac{\partial w_y}{\partial z} \quad (1)$$
where $\vartheta_y$ is the slope of the beam and $\varphi$ is the distortion due to shear. Using the well-known relation between the shear force $f_s$ and $\varphi$ [8]:

$$f_s = kkGA,\varphi$$

(2)

and taking the derivative $\partial / \partial t$ on Equn 1, the following result can be written:

$$\frac{\partial \dot{w}_x}{\partial z} + \dot{\vartheta}_y + \frac{1}{kkGA} \frac{\partial f_s}{\partial t} = 0$$

(3)

in the notation $kk$ symbolizes the Timoshenko shear constant.

The relation between the bending moment $m_y$ and the slope $\vartheta_y$ is given by the following equation of solid mechanics:

$$\frac{\partial \dot{\vartheta}_y}{\partial z} + \frac{1}{EI} \frac{\partial m_y}{\partial t} = 0$$

(4)

Using the notation $M = \rho_iA_i + \rho_sA_s$ for the mass per unit length [9], the equilibrium of the forces in the x-direction yields:

$$-f_x - F_{ox} dz + \left( f_x + \frac{\partial f_x}{\partial z} dz \right) + M dz \frac{\partial \dot{w}_x}{\partial t} = 0$$

(5)

or

$$\frac{\partial f_x}{\partial z} + M \frac{\partial \dot{w}_x}{\partial t} = F_{ox}$$

(6)

The equilibrium of moments about the y-axis with respect to the right end of the pipe element (Fig.1) yields:

$$m_y - \left( m_y + \frac{\partial m_y}{\partial z} dz \right) - M_{oz} dz + (\rho_iI_i + \rho_sI_s) \frac{\partial \dot{\vartheta}_y}{\partial t} - f_s dz = 0$$

(7)

or

$$\frac{\partial m_y}{\partial z} - f_s + B \frac{\partial \dot{\vartheta}_y}{\partial t} = M_{oz}$$

(8)

in which the notation $B$ symbolizes the parameter $B = \rho_iI_i$.

**Flexure in y-z plane**

Following the same procedure, the motion equations for flexure in the y-z plane are:

$$\frac{\partial \dot{w}_y}{\partial z} + \dot{\vartheta}_x + \frac{1}{kkGA} \frac{\partial f_x}{\partial t} = 0$$

(9)

$$\frac{\partial \dot{\vartheta}_x}{\partial z} + \frac{1}{EI} \frac{\partial m_x}{\partial t} = 0$$

(10)

$$\frac{\partial f_y}{\partial z} + M \frac{\partial \dot{w}_y}{\partial t} = F_{ox}$$

(11)

$$\frac{\partial m_x}{\partial z} - f_y + B \frac{\partial \dot{\vartheta}_x}{\partial t} = M_{ox}$$

(12)
Applying the Laplace transform operator $L$ to Eqns 3, 4, 6, 8, and 9-12, the following results are obtained:

\[
\frac{\partial w^*_x}{\partial z} + \dot{\theta}_y + \frac{1}{kGA_y} \left( \dot{f}_x^* - f_x(z,0) \right) = 0
\]  
(13)

\[
\frac{\partial \dot{\theta}_y}{\partial z} + \frac{1}{EI_y} \left( \dot{s} m^*_y - m_y(z,0) \right) = 0
\]  
(14)

\[
\frac{\partial f_x^*}{\partial z} + M \left( \dot{s} \dot{w}_x^* - \dot{w}_x(z,0) \right) = F_{ex}^*
\]  
(15)

\[
\frac{\partial m^*_y}{\partial z} - f_x^* + B \left( \dot{s} \dot{\theta}_y - \dot{\theta}_y(z,0) \right) = M_{ey}^*
\]  
(16)

\[
\frac{\partial w^*_y}{\partial z} + \dot{\theta}_y + \frac{1}{kGA_y} \left( \dot{s} f_x^* - f_x(z,0) \right) = 0
\]  
(17)

\[
\frac{\partial \dot{\theta}_y}{\partial z} + \frac{1}{EI_y} \left( \dot{s} m^*_y - m_y(z,0) \right) = 0
\]  
(18)

\[
\frac{\partial f_x^*}{\partial z} + M \left( \dot{s} \dot{w}_y^* - \dot{w}_y(z,0) \right) = F_{ey}^*
\]  
(19)

\[
\frac{\partial m^*_y}{\partial z} - f_y^* + B \left( \dot{s} \dot{\theta}_x - \dot{\theta}_x(z,0) \right) = M_{ex}^*
\]  
(20)

Decoupling of the governing equations

Adopting the assumption that the spatial distribution of the variables is constant for $t = 0$, i.e. their derivatives with respect to $z$ are zero, the governing Eqns 3, 4, 6, 8, and 9-12 can be decoupled. Following the procedure described in Ref.9, the decoupled equations are summarized in the following list:
\[
\begin{align*}
\frac{\partial^4 \dot{y}}{\partial z^4} - s^2 \left( \frac{B}{EI_t} + \frac{M}{kGA_y} \right) \frac{\partial^2 \dot{y}}{\partial z^2} + \frac{Ms^2}{EI_t} \left( 1 + \frac{Bs^2}{kGA_y} \right) \dot{y} - \frac{MBs^3}{EI_t kGA_y} \dot{y}_y(z,0) \\
+ \frac{Ms^2}{EI_t kGA_y} \dot{f}_y(z,0) &= \frac{s}{EI_t} \frac{\partial F_{ey}}{\partial z} + \frac{s}{EI_t} \frac{\partial^2 M_{ey}}{\partial z^2} + \frac{Ms^3}{EI_t kGA_y} M_{ey}^* \\
\frac{\partial^4 f}{\partial z^4} - s^2 \left( \frac{B}{EI_t} + \frac{M}{kGA_y} \right) \frac{\partial^2 f}{\partial z^2} + \frac{Ms^2}{EI_t} \left( 1 + \frac{Bs^2}{kGA_y} \right) f - \frac{BM^3}{kEI_t kGA_y} f_y(z,0) \\
- \frac{BM^3}{EI_t} \hat{y}_y(z,0) &= \frac{s}{EI_t} \frac{\partial F_{ex}}{\partial z} + \frac{s}{EI_t} \frac{\partial^2 M_{ex}}{\partial z^2} + \frac{Ms^3}{EI_t kGA_y} M_{ex}^* \\
\frac{\partial^4 \dot{w}}{\partial z^4} - s^2 \left( \frac{B}{EI_t} + \frac{M}{kGA_y} \right) \frac{\partial^2 \dot{w}}{\partial z^2} + \frac{Ms^2}{EI_t} \left( 1 + \frac{Bs^2}{kGA_y} \right) \dot{w} - \frac{MBs^3}{EI_t kGA_y} \dot{w}_y(z,0) \\
+ \frac{Ms^2}{EI_t kGA_y} \dot{f}_y(z,0) &= \frac{s}{EI_t} \frac{\partial F_{wy}}{\partial z} + \frac{s}{EI_t} \frac{\partial^2 M_{wy}}{\partial z^2} + \frac{Ms^3}{EI_t kGA_y} M_{wy}^* \\
\frac{\partial^4 \dot{M}}{\partial z^4} - s^2 \left( \frac{B}{EI_t} + \frac{M}{kGA_y} \right) \frac{\partial^2 \dot{M}}{\partial z^2} + \frac{Ms^2}{EI_t} \left( 1 + \frac{Bs^2}{kGA_y} \right) \dot{M} - \frac{MBs^3}{EI_t kGA_y} \dot{M}_y(z,0) \\
+ \frac{Ms^2}{EI_t kGA_y} \dot{f}_y(z,0) &= \frac{s}{EI_t} \frac{\partial F_{Mw}}{\partial z} + \frac{s}{EI_t} \frac{\partial^2 M_{Mw}}{\partial z^2} + \frac{Ms^3}{EI_t kGA_y} M_{Mw}^* \\
\frac{\partial^4 \dot{m}}{\partial z^4} - s^2 \left( \frac{B}{EI_t} + \frac{M}{kGA_y} \right) \frac{\partial^2 \dot{m}}{\partial z^2} + \frac{Ms^2}{EI_t} \left( 1 + \frac{Bs^2}{kGA_y} \right) \dot{m} - \frac{MBs^3}{EI_t kGA_y} \dot{m}_y(z,0) \\
+ \frac{Ms^2}{EI_t kGA_y} \dot{f}_y(z,0) &= \frac{s}{EI_t} \frac{\partial F_{Mw}}{\partial z} + \frac{s}{EI_t} \frac{\partial^2 M_{Mw}}{\partial z^2} + \frac{Ms^3}{EI_t kGA_y} M_{Mw}^*
\end{align*}
\]
Above eight equations can be written in a matrix format as follows:

\[
\begin{bmatrix}
    I_{xss} & \frac{\partial^4}{\partial z^4} \{X'(z, s)\} + \frac{\partial^2}{\partial z^2} \{X'(z, s)\} + [\Lambda(s)] \{X'(z, s)\} = \\
    -[K(s)] \{X(0, s)\} + \{E_r(z, s)\}
\end{bmatrix}
\]

(29)

where

\[
\{X'(z, s)\} = \begin{bmatrix}
    \delta'_x & f'_x & \bar{w}_x & m'_x & \delta'_y & f'_y & \bar{w}_y & m'_y
\end{bmatrix}
\]

(30)

\[
[\Xi(s)] = \begin{bmatrix}
    [B_1(s)] \\
    [O_{4x4}] \\
    [B_4(s)]
\end{bmatrix}
\]

(31)

\[
[\Lambda(s)] = \begin{bmatrix}
    [B_2(s)] \\
    [O_{4x4}] \\
    [B_4(s)]
\end{bmatrix}
\]

(32)

\[
[K(s)] = \begin{bmatrix}
    [B_3(s)] \\
    [O_{4x4}] \\
    [B_4(s)]
\end{bmatrix}
\]

(33)

\[
[B_4(s)] = \left( \frac{B}{EI_l} + \frac{M}{kkGA_y} \right) s^3 [I_{4x4}]
\]

(34)

\[
[B_3(s)] = \left( \frac{M}{EI_l} s^2 + \frac{MB}{EI_lkkGA_y} s^4 \right) [I_{4x4}]
\]

(35)

\[
[B_2(s)] = \left[ \begin{array}{cccc}
    -\frac{B}{kkGA_y} s^3 & \frac{1}{kkGA_y} s^2 & 0 & 0 \\
    -Bs^2 & -\frac{B}{kkGA_y} s^3 & 0 & 0 \\
    0 & 0 & -s & \frac{B}{kkGA_y} s^3 \\
    0 & 0 & 0 & -s - \frac{B}{kkGA_y} s^3
\end{array} \right]
\]

(36)

\[
\{E_r(z, s)\} = \begin{bmatrix}
    \Delta_1 \\
    \Delta_2 \\
    \Delta_3 \\
    \Delta_4 \\
    \Delta_5 \\
    \Delta_6 \\
    \Delta_7 \\
    \Delta_8
\end{bmatrix}
\]

(37)

\[
\Delta_1(z, s) = \frac{1}{EI_l} s \frac{\partial F_{ex}'}{\partial z} + \frac{1}{EI_l} s \frac{\partial^2 M_{ey}'}{\partial z^2} + \frac{M}{EI_lkkGA_y} s^3 M_{ey}'
\]

(38)

\[
\Delta_2(z, s) = \frac{\partial^3 F_{ex}'}{\partial z^3} - \frac{B}{EI_l} s^2 \frac{\partial F_{ex}'}{\partial z} + \frac{M}{EI_l} s^2 M_{ey}'
\]

(39)

\[
\Delta_3(z, s) = -\frac{1}{EI_l} s \frac{\partial M_{ey}'}{\partial z} + \left( \frac{1}{EI_l} s + \frac{B}{EI_lkkGA_y} s \right) F_{ex}'
\]

(40)
\[ \Delta_1(z,s) = \frac{\partial^3 M''_z}{\partial z^3} - \frac{\partial^2 F''_{zy}}{\partial z^2} - \frac{M}{kkGA_s} s^2 \frac{\partial M''_{zy}}{\partial z} \]  
(41)

\[ \Delta_2(z,s) = \frac{\partial^3 M''_z}{\partial z^3} - \frac{\partial^2 F''_{zy}}{\partial z^2} - \frac{M}{kkGA_s} s^2 \frac{\partial M''_{zy}}{\partial z} \]  
(42)

\[ \Delta_3(z,s) = \frac{\partial^3 F''_{zy}}{\partial z^3} - \frac{B}{EI_s} s^2 \frac{\partial F''_{zy}}{\partial z} + \frac{M}{EI_s} s^2 M''_{zy} \]  
(43)

\[ \Delta_4(z,s) = -\frac{1}{EI_s} s \frac{\partial M''_{zy}}{\partial z} + \left( \frac{1}{EI_s} s + \frac{B}{EI_s kkGA_s} s^3 \right) F''_{zy} \]  
(44)

\[ \Delta_5(z,s) = \frac{\partial^3 M''_z}{\partial z^3} - \frac{\partial^2 F''_{zy}}{\partial z^2} - \frac{M}{kkGA_s} s^2 \frac{\partial M''_{zy}}{\partial z} \]  
(45)

**Solution of the dynamic model**

The dynamic response of the pipeline is modelled by Equn 29. The following finite sine Fourier transform:

\[ \left\{ \mathbf{X}'(j,s) \right\} = \int_0^L \left\{ \mathbf{X}'(z,s) \right\} \sin \frac{j\pi z}{L} \, dz \quad j = 1, 2, 3, \ldots \]  
(46)

seems to be an effective tool for the solution of Equn 29. With the aid of the following properties [17]:

\[ \int_0^L \frac{\partial^4 \left\{ \mathbf{X}'(z,s) \right\}}{\partial z^4} \sin \frac{j\pi z}{L} \, dz = \frac{j^4 \pi^4}{L^4} \left\{ \mathbf{X}'(j,s) \right\} \]  
(47)

\[ \int_0^L \frac{\partial^2 \left\{ \mathbf{X}'(z,s) \right\}}{\partial z^2} \sin \frac{j\pi z}{L} \, dz = -\frac{j^2 \pi^2}{L^2} \left\{ \mathbf{X}'(j,s) \right\} \]  
(48)

\[ \int_0^L \sin \frac{j\pi z}{L} \, dz = \frac{L}{j\pi} (1 - \cos j\pi) \]  
(49)

Equn 29 yields:

\[ \left[ I_{s,s} \right] \frac{j^4 \pi^4}{L^4} \left\{ \mathbf{X}'(j,s) \right\} - \left[ \Xi(s) \right] \frac{j^2 \pi^2}{L^2} \left\{ \mathbf{X}'(j,s) \right\} + \left[ \Lambda(s) \right] \left\{ \mathbf{X}'(j,s) \right\} = -\left[ K(s) \right] \{ x(j) \} + \{ e_i(j,s) \} \]  
(50)

where

\[ \{ x(j) \} = \int_0^L \{ X(z,0) \} \sin \frac{j\pi z}{L} \, dz \]  
(51)
\[ \{e_j(j,s)\} = \frac{L}{\pi} \int_0^L \{E_j(z,s)\} \sin \frac{j\pi z}{L} dz \]  

(52)

Equation 50 is a simple algebraic equation with respect to the double-transformed variable, solution of which yields:

\[ \{X^*(j,s)\} = \left( \frac{j^2 \pi^2}{L^2} [I_{sxx}] - \frac{j^2 \pi^2}{L^2} [\Xi(s)] + [\Lambda(s)] \right)^{-1} \left( -[K(s)](x(j)) + \{e_j(j,s)\} \right) \]

(53)

Inversion of the finite sine Fourier transform of the above equation results in:

\[ \{X^*(z,s)\} = \frac{2}{L} \sum_{j=1}^\infty \left( \frac{j^2 \pi^2}{L^2} [I_{sxx}] - \frac{j^2 \pi^2}{L^2} [\Xi(s)] + [\Lambda(s)] \right)^{-1} \left( -[K(s)](x(j)) + \{e_j(j,s)\} \right) \sin \frac{j\pi z}{L} \]

(54)

Then, taking the inverse Laplace transform in above equation, the final solution can be obtained:

\[ \{X(z,t)\} = \frac{2}{L} \sum_{j=1}^\infty \{\Psi(j)\} \sin \frac{j\pi z}{L} \]

(55)

where

\[ \{\Psi\} = L^{-1} \left( \left( \frac{j^2 \pi^2}{L^2} [I_{sxx}] - \frac{j^2 \pi^2}{L^2} [\Xi(s)] + [\Lambda(s)] \right)^{-1} \left( -[K(s)](x(j)) + \{e_j(j,s)\}\right); s \to t \right) \]

(56)

**Numerical examples**

A modern methodology for pipeline inspection is based on the transmission of guided waves. The wave generation is carried out by impact loads, and therefore the wave propagation in a pipeline under impact flexural load:

\[ F_{ex}^{imp} = F_0 \frac{1}{\sqrt{\pi}} e^{-z(a)^2} \delta(t) \]

(57)

will be modelled with the aid of the solution given by Eqn 55. In Eqn 57, the function

\[ f(z) = \frac{1}{\sqrt{\pi}} e^{-z(a)^2} \]

(58)

demonstrates the spatial distribution of the applied flexural load \( F_{ex} \), and the Dirac delta function \( \delta(t) \) simulates the impact loading. For \( a = 2 \) the load \( F_{ex} \) lies on the area \(-5.0 \text{m} \leq z \leq 5.0 \text{m}\) (Fig.2).

Apart from impact loading, a harmonic loading is another usual case in engineering practice. Since pumps and compressors are usual parts of pipelines, harmonic loads of the following type

\[ F_{ex}^{har} = F_0 \frac{1}{\sqrt{\pi}} e^{-z(a)^2} \sin \omega t \]

(59)
are causing flexural vibrations on the pipeline. With the aid of Equn 55, the dynamic response of a pipeline subjected to (a) impact loading given by Equn 57, and (b) harmonic loading given by Equn 59, will be obtained. The considered pipeline has a length \( L = 100 \) m. Its inner radius and the wall thickness are 30 m and 4.0 mm, respectively. The elasticity modulus and the Poisson’s ratio of the pipeline material (steel) are \( E = 200 \) GPa and \( \nu = 0.3 \). The density of the steel is \( \rho_t = 7990 \) kg/m\(^3\), the density of the liquid is \( \rho_l = 7990 \) mg/m\(^3\), and the bulk modulus of elasticity is \( K_l = 2.14 \times 10^9 \) Pa. The selected value for the load amplitude is \( F_0 = 10^9 \) N.

Figure 3 demonstrates the shear force \( f_x \) distribution along the pipeline for \( t = 0.0005 \) sec, 0.0100 sec, and 0.0150 sec due to the impact flexural load given by Equn 57.

For the harmonic load of Equn 59, the dynamic response of the pipeline is shown in Fig.4.

Figure 3 indicates that the wave amplitude travels along the pipeline, and the maximum values of the wave take place near the location of the loading source. The first peak of the wave travels a distance of \( \Delta z = 1.615 \) m within a time \( \Delta t = 0.0140.005 = 5 \times 10^{-3} \) sec. Therefore, the wave velocity is 323 m/sec.

Conclusions

- A model simulating the vibration response of a pipeline under external flexural loading has been derived.
- A set of eight coupled first-order differential equations have been decoupled, yielding a system of eight fourth-order partial differential equations.
- An analytical solution based on the finite sine Fourier transform and the Laplace transform has been developed.
- Representative examples of a pipeline subjected to impact and
harmonic external loads have been provided.

References

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